# Evolving Bunch and Retardation in the Impedance Formalism

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- Motivation for impedance rather than wake potential (or its integral) to compute the collective force of CSR.

- Complete impedance $Z(n,\omega)$ versus its "diagonal part" $Z(n)=Z(n,n\omega_0)$. Required, in principle, when the bunch profile evolves in time.

- General form of CSR force and radiated power. Causality and retardation.

- Practical computation of force and power in a Vlasov or macroparticle simulation.

# Impedances and Wakes in High-Energy Particle Accelerators


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- Practical computation of force and power (cont'd):

Reduction of $\sum_n (\cdots) \int d\omega (\cdots)$ can be done in terms of $Z(n, n\omega_0)$ and $\partial Z/\partial\omega(n, n\omega_0)$, to a good approximation, except for $\omega$ near waveguide cutoffs

$$\omega_p = \pm \frac{\pi p c}{h} \ , \quad p = 1, 3, \cdots \ ,$$

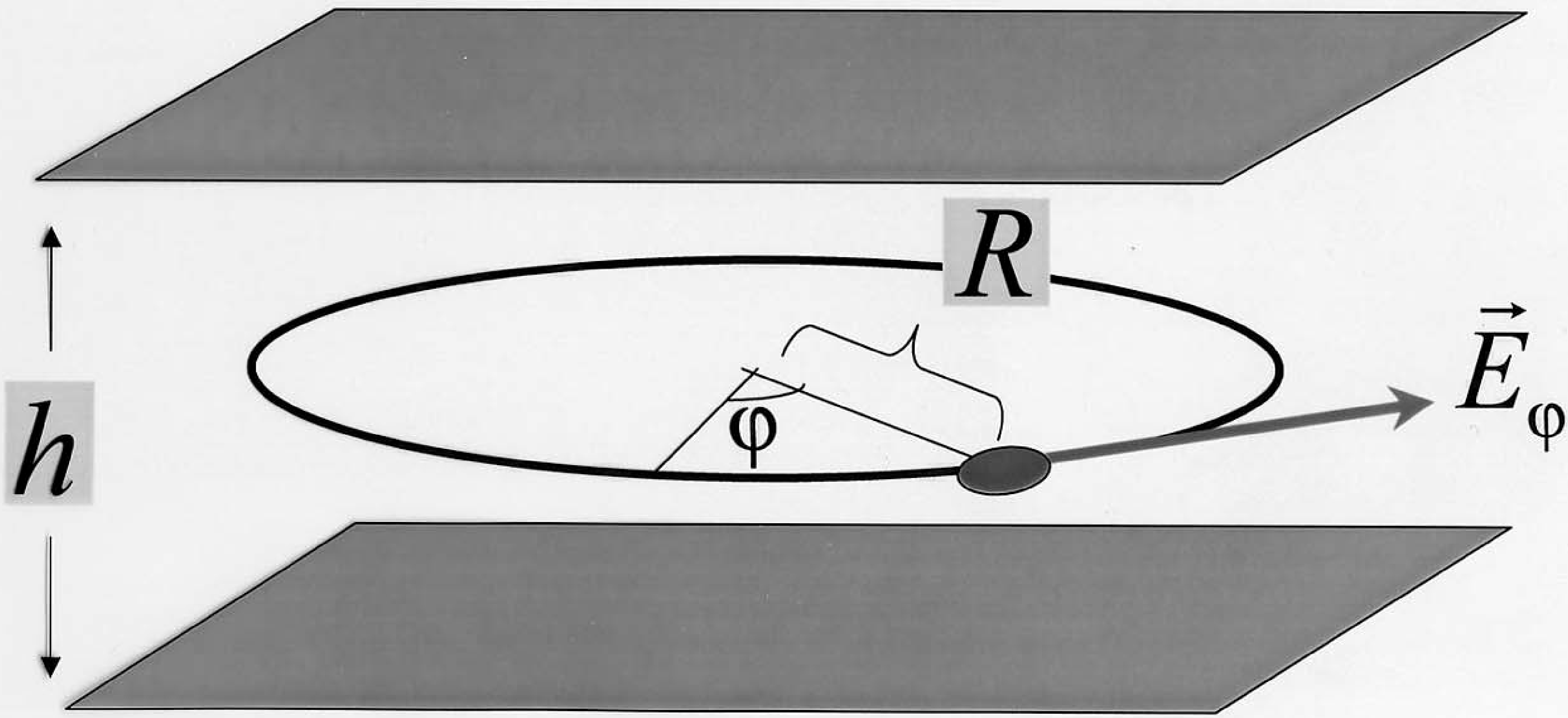
where $Z(n, \omega)$ has a pole in $\omega$. Pole is associated with prominent retardation effects.

## Shielding Model

For dynamical studies it is essential to including shielding of CSR due to the vacuum chamber. For this talk, assume parallel-plate model, plate separation $h$, source on circular orbit of fixed radius $R$ in mid-plane. Cylindrical coordinates $(r, \theta, y)$, with $y$ perpendicular to plates.

Our story can be adapted to other models with analytic solutions, but maybe not to models solved numerically (Stupakov & Kotelnikov).

# Parallel Plate Model for CSR:
## Geometry Outline



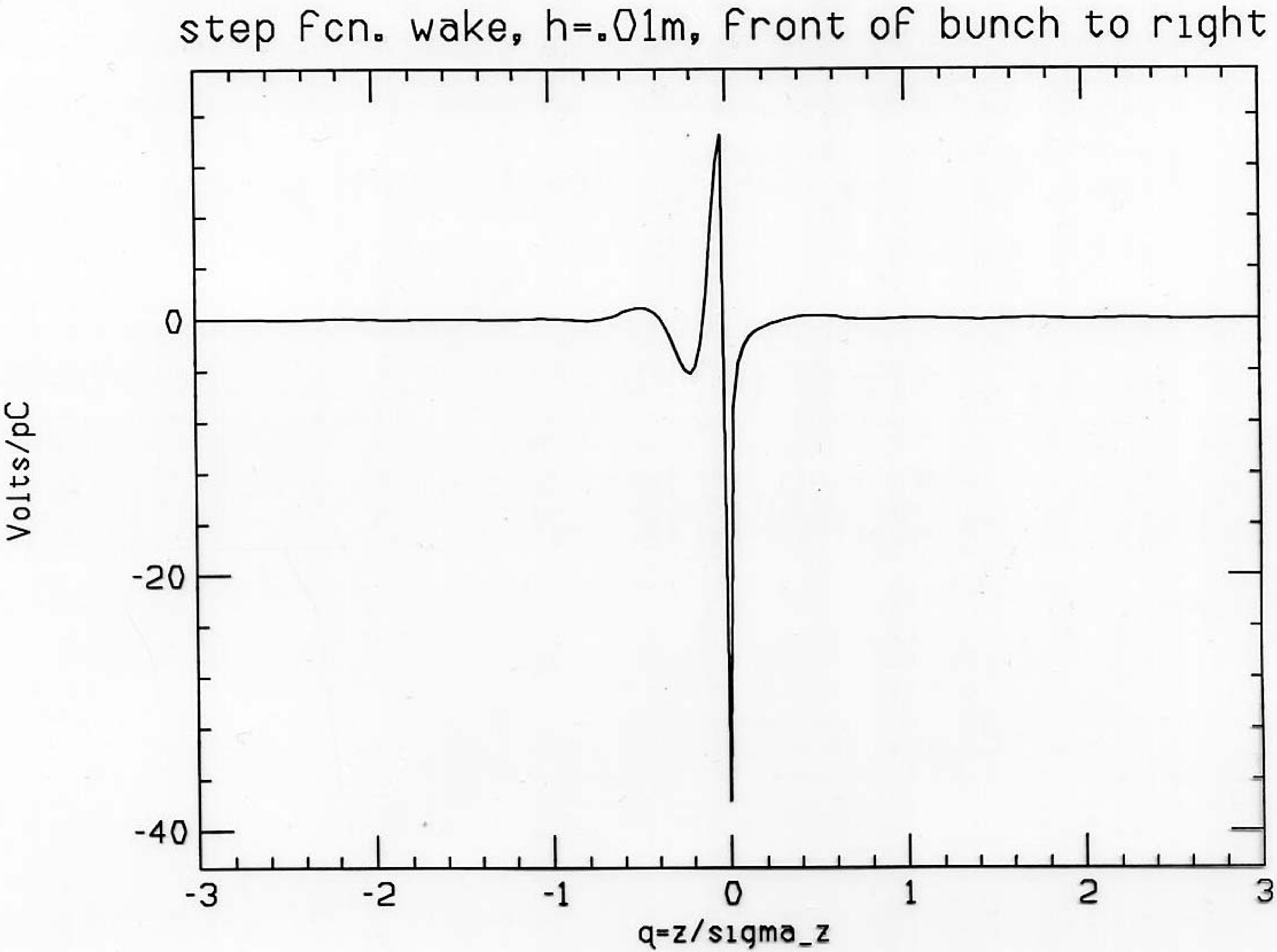
## Impedance vs. Wake Potential

Conventionally, the wake voltage can be represented by either the impedance $Z(n)$, or the wake potential $W(z)$, or the integral $S(z)$ of $W$:

$$V(z) = Q\omega_0 \sum_n e^{inz/R} Z(n)\lambda_n =$$

$$Q \int W(z-z')\lambda(z')dz' = -Q \int S(z-z')\lambda'(z')dz'$$

For our radiation impedance, even $S(z)$ is too concentrated at small $z$ to be useful in a Vlasov simulation. We can use $Z(n)$ successfully, with the sum on $n$ converging quickly by virtue of the fall-off of $\lambda_n$.


step fcn. wake, h=.01m, front of bunch to right
Volts/pC
0
-20
-40
-3
-2
-1
0
1
2
3
q=z/sigma_z

## Laplace Transform of Current

Longitudinal current has the form

$$I(\theta,t) = Q\omega_0\lambda(\theta-\omega_0 t,t)\ ,\quad \lambda(\theta,t)=0\ ,\ t\le 0\ ,$$

which has a Fourier transform with Im$\omega > 0$ (equivalent to Laplace transform)

$$\begin{aligned}\hat{I}(n,\omega) &= \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-in\theta}d\theta\int_{-\infty}^{\infty} e^{i\omega t}I(\theta,t)\\ &= \frac{Q\omega_0}{2\pi}\int_0^{\infty} e^{i(\omega-n\omega_0)t}\lambda_n(t)dt\ ,\quad \mathrm{Im}\omega = v > 0.\end{aligned}$$

Assume that $\lambda_n \in C^p\ ,\ p\ge 2$ with
$\lambda_n^{(k)}(0)=0\ ,\ k=0,1,\cdots,p-1.$

## Wake Voltage in Terms of Complete Impedance

Taking similar Laplace transform of Maxwell's equations and solving for the longitudinal electric field $\hat{\mathcal{E}}(n,\omega)$, we define $Z(n,\omega)$ by

$$-2\pi R\hat{\mathcal{E}}(n,\omega) = Z(n,\omega)\hat{I}(n,\omega) = \hat{V}(n,\omega)\ .$$

Hence the general form of the wake voltage is

$$V(\theta,t) = \\ Q\omega_0\sum_n e^{in\theta}\int_{\mathrm{Im}\omega=v} d\omega e^{-i\omega t} Z(n,\omega) \\ \cdot\frac{1}{2\pi}\int_0^\infty e^{i(\omega-n\omega_0)t'}\lambda_n(t')dt'\ .$$

## Radiated Power in Terms of Complete Impedance

$$P(t) =$$

$$(Q\omega_0)^2 \sum_n e^{in\theta} \lambda_n(t) \int_{\mathrm{Im}\omega = v} d\omega e^{-i\omega t} Z(n,\omega)$$

$$\cdot \frac{1}{2\pi} \int_0^\infty e^{i(\omega - n\omega_0)t'} \lambda_n(t') dt' \ .$$

Depends on both Re$Z$ and Im$Z$.

## Causality and Analyticity

Contribution of $\lambda_n(t)$ for $t > 0$ should vanish by causality. Mathematically, this happens because $Z(n,\omega)$ is analytic in $\omega$ in the upper half-plane, and obeys $|Z(n,\omega)| \leq M \;, \quad \mathrm{Im}\omega \geq 0$. Integrating once by parts we can get the bound

$$\left| \int_{t+\delta t}^{\infty} e^{i(\omega - n\omega_0)t'} \lambda_n(t') dt' \right| \leq \frac{M}{|\omega - n\omega_0|} e^{-\mathrm{Im}\omega(t+\delta t)} \;.$$

This shows that when the $\omega$-contour is moved to a semi-circle at infinity in the upper half-plane, the contribution of $\int_{t+\delta t}^{\infty}$ vanishes for any $\delta t > 0$.

## The Limit $\delta t \to 0$

For any $\delta t > 0$ we have

$$V(\theta, t) = \frac{Q\omega_0}{2\pi} \sum_n e^{in\theta} \int_{\mathrm{Im}\omega = v} d\omega Z(n, \omega) \int_0^{t+\delta t} dt' e^{i(\omega - n\omega_0)t'} \lambda_n(t') \ .$$

Can we put $\delta t = 0$ in this equation? Strangely enough, the answer is NO! The $\omega$-integral does not converge uniformly w.r.t. $\delta t$, so taking the limit $\delta t \to 0$ under the integral is not justified, and in fact gives the wrong answer.

## Strategy for the Limit $\delta t \to 0$

Integrate twice by parts on $t'$ to get inverse powers of $\omega - n\omega_0$. Then the $\omega$-integral converges uniformly and we can take the limit under the integral. Then integrate by parts in opposite direction. The $\omega$-integral becomes

$$-\int d\omega e^{-i\omega t} \frac{Z(n,\omega)}{2\pi(\omega - n\omega_0)^2} \int_0^t dt' e^{i(\omega - n\omega_0)t'} \lambda_n''(t')$$

$$= i \int d\omega e^{-i\omega t} \frac{Z(n,\omega)}{2\pi(\omega - n\omega_0)} \int_0^t dt' e^{i(\omega - n\omega_0)t'} \lambda_n'(t') \ .$$

## One More Integration by Parts Raises Hell!

$$\int d\omega e^{-i\omega t}\frac{Z(n,\omega)}{2\pi}\int_0^t dt' e^{i(\omega-n\omega_0)t'}\lambda_n(t')$$
$$-\frac{1}{2\pi i}\lambda_n(t)\int d\omega\frac{Z(n,\omega)}{\omega-n\omega_0}\ .$$

The first term is what we would get by putting $\delta t = 0$ in the original integral. The second term does not exist unless defined as a symmetric limit (which is allowed):

$$\lim_{\Omega\to\infty}\int_{-\Omega+iv}^{\Omega+iv}\frac{d\omega Z(n,\omega)}{\omega-n\omega_0}\ .$$

We can base calculations on the forms with $\lambda_n'$ or $\lambda_n''$.

$$V(\theta,t)=-Q\omega_0\sum_n e^{in\theta}\int d\omega e^{-i\omega t}\frac{Z(n,\omega)}{2\pi(\omega-n\omega_0)^2}$$
$$\cdot\int_0^t dt' e^{i(\omega-n\omega_0)t'}\lambda_n''(t')$$

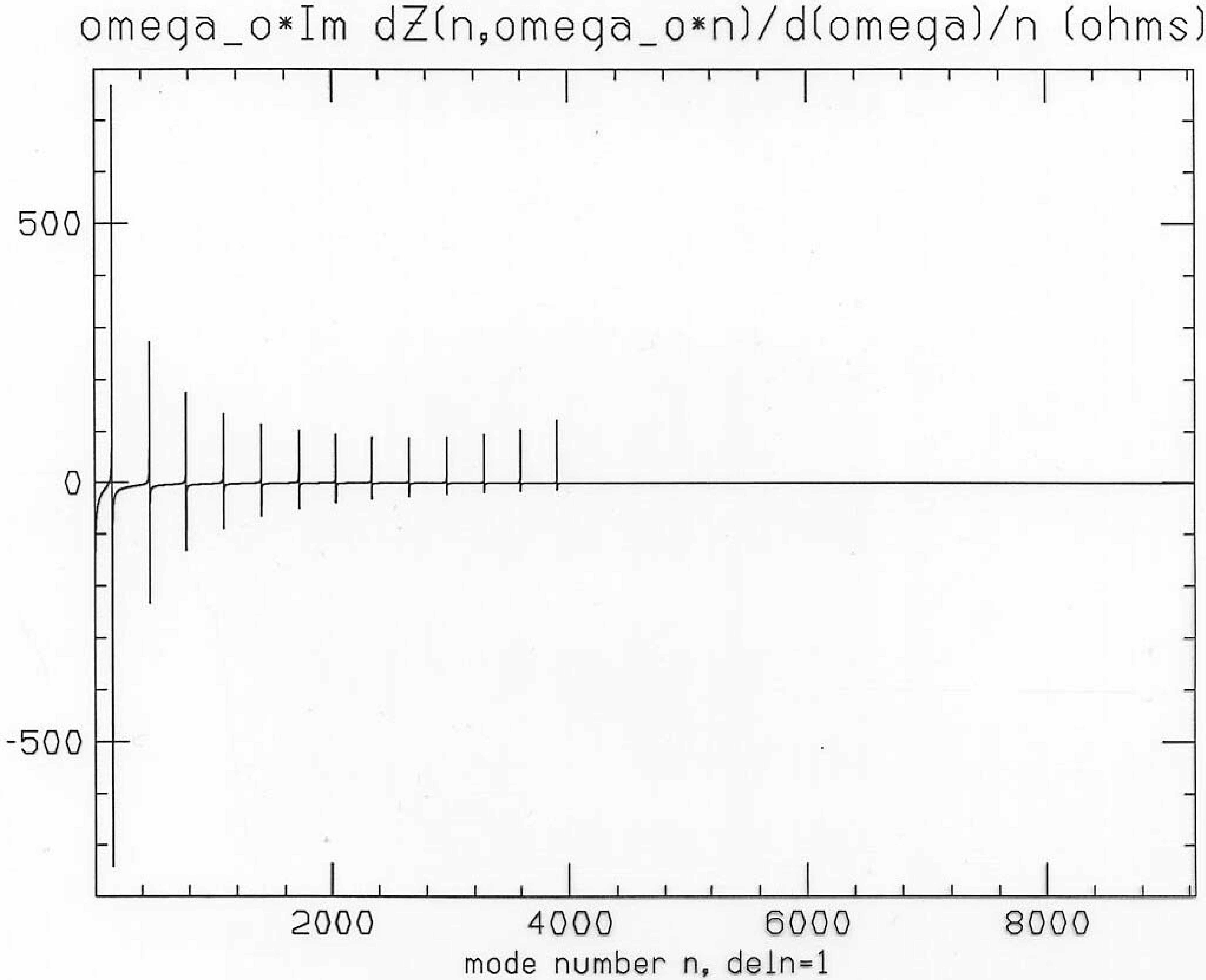
- The 2nd order pole concentrates the $\omega$-integral near $n\omega_0$.
- If $\lambda_n''(t)$ can be regarded as constant over any time interval $\Delta t$ (i.e., $\lambda_n$ is locally quadratic) then the $t'$-integral is proportional to $\mathrm{sinc}((\omega-n\omega_0)\Delta t/2)$, also concentrated near $n\omega_0$.

So let's expand $Z(n,\omega)$ about $n\omega_0$!

Taking two terms and applying residue theorem, we find

$$V(z,t) =$$
$$Q\omega_0 \sum_n e^{inz/R}\Big( Z(n,n\omega_0)\lambda_n(t) + i\frac{\partial Z}{\partial\omega}(n,n\omega_0)\lambda_n'(t)\Big) \ .$$

This does not work near waveguide cutoffs $\omega = \pm\pi p/h$, where $Z(n,\omega)$ has poles. Curiously, the residues of these poles vanish at $n\omega_0 = \pm\pi p/h$, so they do not show up in a plot of $Z(n,n\omega_0)$. They do show up in the derivative.


omega_o*Im dZ(n,omega_o*n)/d(omega)/n (ohms)
500
0
-500
2000
4000
6000
8000
mode number n, deln=1

## Evaluate the Pole Term and Expand the Remainder in a Taylor series

Let $\tilde{Z}$ be the non-pole remainder. Then the result is

$$
\begin{aligned}
&V(z,t) = \\
&Q\omega_0 \sum_n e^{inz/R} \Big[ \tilde{Z}(n, n\omega_0)\lambda_n(t) + i\frac{\partial \tilde{Z}}{\partial \omega}(n, n\omega_0)\lambda_n'(t) + \cdots \\
&+ \frac{Z_0 \pi R}{2\beta h} \sum_p \Lambda_p \int_{-t}^{0} \lambda_n(t+u)du \Big( (n\omega_0 - \alpha_p c) e^{-i(n\omega_0 - \alpha_p c)} \\
&+ (p \to -p) \Big) \Big] \,, \quad \alpha_p = \pi p / h \;.
\end{aligned}
$$

Retardation associated with waveguide cutoffs.

## Conclusion

- To treat an evolving bunch in the impedance formalism, we have plausible corrections to the naive replacement $\lambda_n \to \lambda_n(t)$, which are not expensive to compute. We avoid an expensive computation of the double sum $\sum_n(\cdots)\int d\omega(\cdots)$.
- We have discovered an interesting retardation effect associated with waveguide cutoffs. One should try to understand this in physical terms.
- Numerical results coming soon!